Chapter 8
Stability Analysis

Introduction

The motor controller is fundamentally based on feed-forward control methods but it uses feedback of the motor currents to provide corrections for feed-forward errors and for load-torque changes and to provide damping. This chapter explores how the feedback correction and damping works and provides the analysis necessary to select the feedback gain constants.

Although there is little change in the feedback configuration with speed, except for some gain constants that ramp up or down as speed increases, the way the feedback works changes as motor speed is increased from zero to maximum.

There are three regions of motor speed through which the effect of feedback changes: low speed, mid speed and high speed. The low-speed region is operation at standstill and low speed where the controller relies on a positive *d*-axis current to lock the rotor into the required position. The second is the mid-speed region, where the controller uses the predominantly resistive stator voltage drops to measure and correct rotor angle errors and to provide damping. The last is the high-speed region where the predominantly inductive stator-voltage drops result in the motor largely self correcting for rotor angle errors with assistance from the feedback and with damping also provided by the feedback.

In this chapter, operation in the high-speed region is analysed first, followed by analysis of the mid-speed region, and finally analysis of the low-speed region. For the high-speed and mid-speed regions, separate analyses are required for speed-control mode and torque-control mode. For the low-speed region, torque control is not possible, so only speed-control mode is analysed.

There are many ways of analysing a controller to determine its response and its stability. Common methods are root-locus analysis, Bode plots, and the Nyquist stability criterion. Here, the more unusual method of representing the controller with equivalent electrical circuits is used. This gives a clearer picture of the controller stability and response when third and higher order systems are being analysed as is the case here. It also allows for the effects on stability of different loads to be easily determined.

High Speed Region

This section investigates the response in the high speed region where motor speeds are well above the natural resonance ω_n and where $\omega L >> R$ (ω: rotor frequency) allowing the motor resistance to be ignored.

For the high speed region analysis it is necessary to consider separately the cases when the controller is operating in torque control mode and in speed control mode. The case when the controller is in torque control mode is considered first.

Motor Dynamics Equivalent Circuit

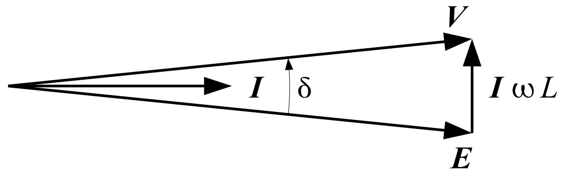

Figure 8.1. Phasor diagram for rotor retarded by angle δ

In the high speed region, for small perturbations, the q axis torque producing current i_q can be assumed to be sensitive to changes in the rotor angle only and can be analysed most simply using phasor diagrams.

Consider the case of a two pole PMSM operating without load at a high fixed stator frequency ω and with the rotor EMF vector E equal to the applied voltage vector V. Now consider the case of a small perturbation of δ radians in the rotor angle due to instability causing a current I to flow. Ignoring motor resistance, which at high speed would normally be much less than the motor reactance ωL anyway, the phasor diagram shown in Figure 8.1 applies.

For a small value of δ, $|\boldsymbol{I}|$ is equal to i_q and with $|\boldsymbol{E}|=\lambda_r \omega$ the resulting current is given by:

$$i_q=\frac{\lambda_r \delta}{L} \tag{8.1}$$

The resulting restoring torque T is given by:

$$\begin{aligned} T &= \lambda_r i_q \\ &= \frac{\lambda_r^2 \delta}{L} \end{aligned} \tag{8.2}$$

This causes the rotor speed to change to reduce δ, restrained by the inertia J. The equation of motion is:

$$\begin{aligned} T &= -J\frac{d\omega}{dt} \\ &= -J\frac{d^2\delta}{dt^2} \end{aligned} \tag{8.3}$$

For the analysis, the effect of a change in the controller speed ω' driving term can be added to equation (3) as follows:

$$T=-J\frac{d^2\delta}{dt^2}+J\frac{d\omega'}{dt} \tag{8.4}$$

Replacing δ with T as the independent variable using equation (8.2) and rearranging:

$$\frac{T}{J} = -\frac{L}{\lambda_r^2}\frac{d^2 T}{dt^2} + \frac{d\,\omega'}{dt} \qquad (8.5)$$

Integrating this assuming zero initial conditions:

$$\frac{1}{J}\int T\,dt = -\frac{L}{\lambda_r^2}\frac{dT}{dt} + \omega' \qquad (8.6)$$

It can be seen from this integration that equation (8.5) can be represented by the mechanical equivalent electrical circuit shown in Figure 8.2 where $L_M = L/\lambda_r^2$ and $C_M = J$ are the mechanical equivalent inductance and capacitance values (inverse of torsional spring constant and moment of inertia). The resonant frequency of this circuit is the natural frequency ω_n ($=\lambda_r/\sqrt{LJ}$). The voltage across C_M represents the rotor speed ω.

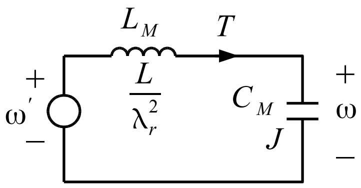

Figure 8.2. High speed quasi-steady-state mechanical equivalent circuit

Feedback Analysis for Torque Control

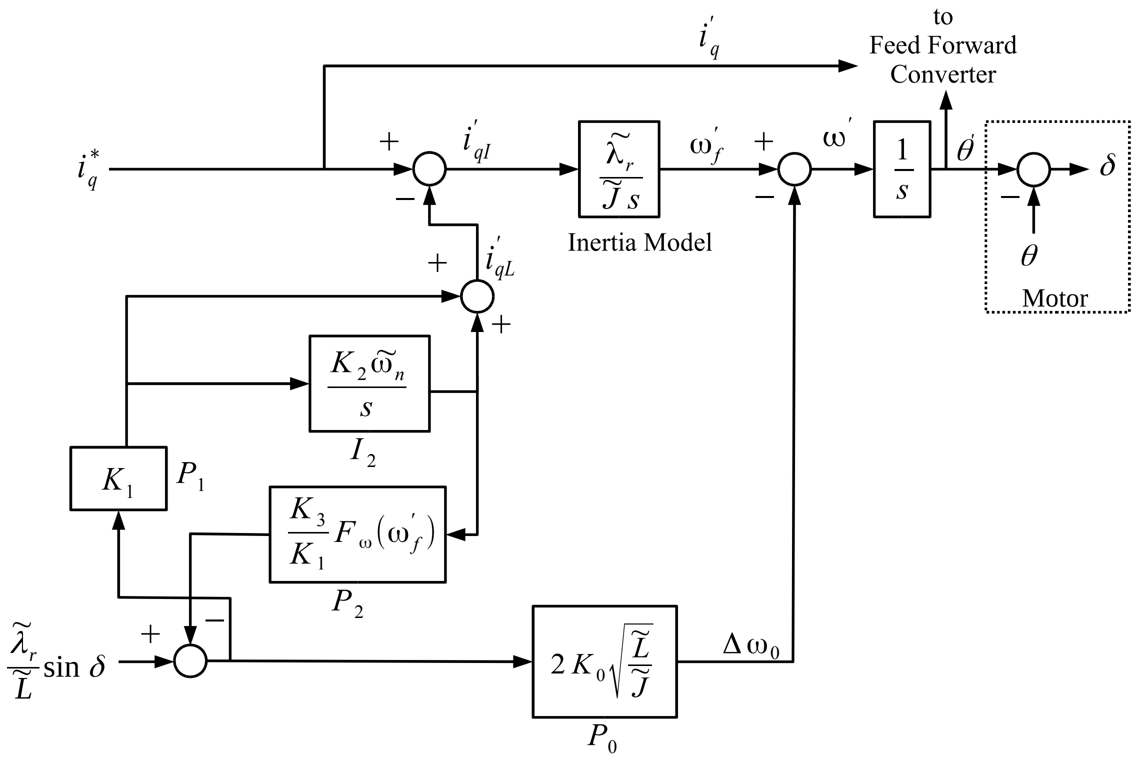

Figure 8.3. Feedback control block diagram under torque control.

Figure 8.3 presents block diagram of the system under torque control derived from Figure 5.1 in Chapter 5. The flux angle error δ is the difference controller output flux angle θ' and the motor flux angle θ. This angle error is measured by the controller using the measured currents as described in Chapter 4. Because a change in θ' results in an instantaneous change in the measured δ and a change in δ has little effect on the motor torque in the mid-speed region, the feedback loop to minimise δ is effectively internal making stability calculations simple.

The torque control structure controls the damping of the mechanical equivalent circuit in Figure 8.2 via control of ω'. The gain constants K_0, K_1 and K_2 should be chosen to obtain adequate damping at the highest possible gain. Ignoring for now the integral block I_2 and its feedback block P_3 which are adjusted to have only a minor effect on the response characteristics, the effect of proportional gain blocks P_0 and P_1 can be incorporated into the high speed equivalent circuit. This will aid in determining the values of constants K_0 and K_1.

Assuming no change in the command torque input, the transfer function of the P_0 and P_1 blocks for a change in the applied speed from a change in the q axis current is given by:

$$\Delta\omega' = \frac{K_1\tilde{\lambda}_r}{\tilde{J}\,s}\Delta i_q - 2K_0\sqrt{\frac{\tilde{L}}{\tilde{J}}}\Delta i_q \tag{8.7}$$

When added to the high speed equivalent circuit, the two terms of equation (8.7) become a mechanical equivalent capacitor and resistor (inertia and inverse of friction). These components and their values are shown in the expanded equivalent circuit of Figure 8.4.

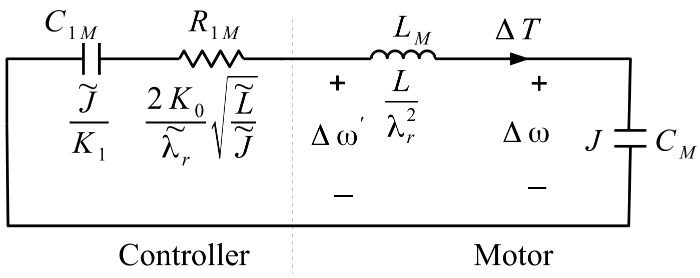

Figure 8.4. High speed equivalent circuit of controller and motor for stability analysis

This equivalent circuit allows easy selection of gain constants K_0 and K_1. Suitable settings for most applications are $K_0 = 1$ setting R_{1M} for critical damping of the series $R_{1M}L_MC_M$ circuit and $K_1 = 0.5$ making C_{1M} twice C_M which is high enough to not interfere too much with circuit damping. These values provide a good compromise between compensator gain and damping. Testing has shown that if maximum response time is not required, $K_1 = 0.25$ is a better choice as it reduces controller noise.

The gain constant K_2 of the second order integral I_2 in Figure 5.1 is set to as high as possible without interfering too much with the response characteristic provided by the PI compensator. A value of $K_2 = K_1$ was found by experiment to be the best compromise when operating in the high speed region. For mid-speed operation, $K_2 = 0.5\,K_1$ was found to be the best compromise with the higher value causing unwanted damped oscillatory response.

Of interest is the effect of a change in the torque command T^*, which is the torque command current times the rotor flux $i_q^* \times \lambda_r$, on the equivalent circuit of Figure 8.4. This is shown in Figure 8.5. The torque injection shown on the left is the effect of a change in the torque current input i^*_q shown in Figure 5.1 in Chapter 5. The torque injection shown on the right is the effect of the Feed Forward Converter changing the motor current as a result of a change in i^*_q, assuming the feed forward motor parameters match the actual motor parameters. A change in the command torque has an immediate effect on the motor torque without exciting the equivalent tuned circuit. Note that any parameter mismatch will result in an effective load torque disturbance resulting in the same damped oscillation response. This makes the controller fairly robust to parameter errors.

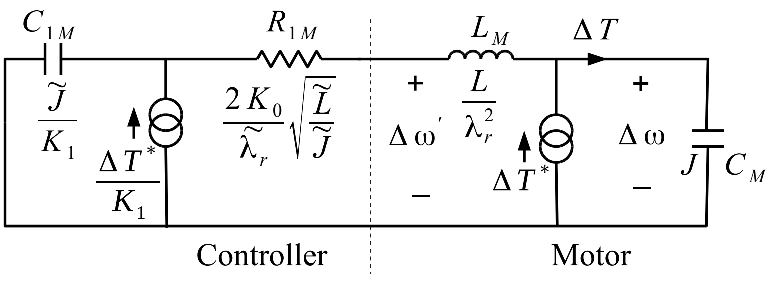

Figure 8.5. High speed equivalent circuit showing the effect of a change in command torque

Feedback Analysis for Speed Control

Now consider the case when the controller is operating in speed control mode. The local speed control loop gives direct control of the filtered speed ω'_f bypassing the torque PI compensator. The resulting controller and motor equivalent circuit is shown in Figure 8.6 to a first approximation. Shown dotted is the effect of a load torque change ΔT_L. Also shown in this figure is the effect of a change in ω'_f. A change in ω'_f is accompanied with a torque current pulse with a resulting torque pulse $\Delta T'$ which causes the rotor speed to exactly follow ω'_f provided the estimated motor parameters are correct. Again, the effect of the integral block I_2 is ignored for this analysis. Its effect is to speed up the response and reduce damping slightly.

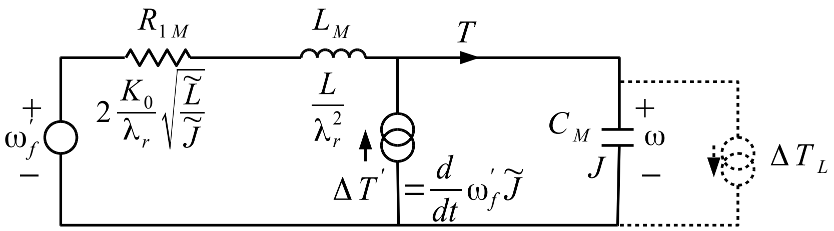

Figure 8.6. High speed equivalent circuit in speed control mode.

It is of interest to investigate the effect the controller has on a load connected with a high compliance coupling causing a low frequency mechanical resonance. In a conventional controller the command torque signal must be filtered to avoid exciting this resonance. The effect of adding a two-mass coupled load on the new controller can be seen by adding the load's equivalent mechanical circuit to the high speed equivalent circuit. This extension is shown in Figure 8.7. Capacitances C_M and C_L are the mechanical equivalent of the motor and load inertias and inductance L_C is equivalent coupling compliance. This circuit shows that by adjustment of the controller gains, particularly K_0, it should be possible to provide at least partial damping of the load coupling resonance, which is usually sufficient to stop oscillations.

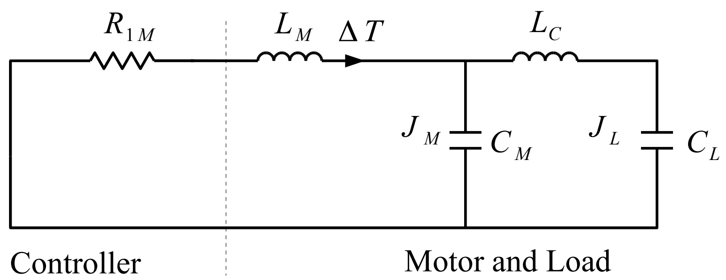

Figure 8.7. System equivalent circuit of damping for a compliantly coupled load.

Mid Speed Region

First step: include effect of resistance in the equivalent circuit of Figure 8.2.

Motor Dynamics Equivalent Circuit

In the mid speed region, the effect of motor resistance becomes significant compared to the high speed region. With motor resistance included, the phasor diagram of Figure 8.1 for when the motor is supplied by a rotating voltage of amplitude equal to the back emf becomes that of Figure 8.8.

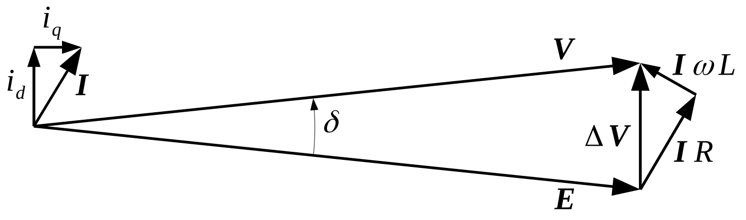

Figure 8.8. Phasor diagram for rotor retarded by angle δ with resistance.

The phasor diagram, which is only approximate, shows the currents and voltages when the rotor lags the applied voltage by a small angle δ. The resulting *d* axis current i_d has no effect on the motor operation apart from generating heat. The resulting torque is dFeedback Analysis for Torque Controletermined by the *q* axis current i_q only.

Using complex circuit analysis, the circuit for calculating i_q is shown in Figure 8.9.

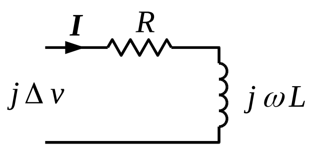

Figure 8.9. Circuit for calculating currents.

From this circuit:

$$\boldsymbol{I}(R+j\,\omega L)=j\,\Delta v$$
$$\boldsymbol{I}=\Delta v\frac{\omega L}{R^2+(\omega L)^2}\;+\;j\,\Delta v\frac{R}{R^2+(\omega L)^2} \tag{8.8}$$

This gives i_q as:

$$i_q=\Delta v\frac{\omega L}{R^2+(\omega L)^2} \tag{8.9}$$

For a rotor flux λ_R, the resulting torque T ($=i_q\lambda_R$) is:

$$T=\Delta v\,\lambda_R\frac{\omega L}{R^2+(\omega L)^2} \tag{8.10}$$

For small δ in radians:

$$\begin{aligned}\Delta v &= \delta E \\ &= \delta \lambda_R \omega\end{aligned} \tag{8.11}$$

Using equations 8.10 and 8.11, δ can be found in terms of T as:

$$\delta = T \frac{L}{\lambda_R^2}\left(1+\frac{R^2}{(\omega L)^2}\right) \tag{8.12}$$

This assumes zero initial torque and current, but for for non-zero initial conditions, equation 8.12 still holds for small values of δ giving the following relation:

$$\delta = T \frac{L}{\lambda_R^2}\left(1+\frac{R^2}{(\omega L)^2}\right) \tag{8.13}$$

With the effect of motor resistance included, the high speed mechanical equivalent circuit of Figure 8.2 becomes the circuit of Figure 8.10

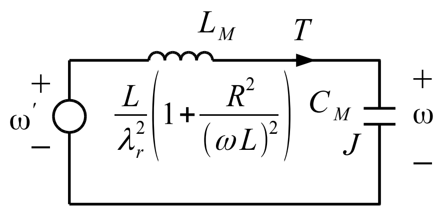

Figure 8.10. Mechanical equivalent circuit including resistance.

Feedback Analysis for Torque Control

It can be seen from Figure 8.10 and also from the phasor diagram of Figure 8.8 that at mid speeds where motor R is much greater than ωL small changes in angle error δ results in negligible changes in torque ΔT. But from the phasor diagram small changes in δ results in large changes in i_d.

Figure 8.3 presents block diagram of the system under torque control derived from Figure 5.1 in Chapter 5 which can be used for mid-speed feedback analysis.

As shown in Figure 8.3 the flux angle error δ is the difference controller output flux angle θ' and the motor flux angle θ. This angle error is measured by the controller using the measured currents as described in Chapter 4. Because a change in θ' results in an instantaneous change in the measured δ and a change in δ has little effect on the motor torque in the mid-speed region, the feedback loop to minimise δ is effectively internal making stability calculations simple.

In the feedback system of Figure 8.3, the P_2 block can be ignored as the function F_ω is zero in the mid-speed region. Also, the I_2 block can be ignored since K_2 is set so that this second order integral gain block has only a minor effect on stability. With these blocks ignored and sin δ approximated to δ in radians, the feedback equation in Laplace notation can be written as:

$$\delta s = -2K_0\sqrt{\frac{L}{J}}.\frac{\lambda_r}{L}\delta - K_1\frac{\lambda_r}{Js}.\frac{\lambda_r}{L}\delta \tag{8.14}$$

Re-arranging as a differential equation of motion:

$$\frac{d^2\delta}{dt^2} + 2K_0.\frac{\lambda_r}{\sqrt{LJ}}\frac{d\delta}{dt} + K_1.\frac{\lambda_r^2}{LJ}\delta = 0 \tag{8.15}$$

Comparing this to the general equation of motion for damped oscillation with resonant frequency ω_0 and damping factor ζ:

$$\frac{d^2\delta}{dt^2} + 2\zeta\omega_0\frac{d\delta}{dt} + \omega_0^2\delta = 0 \tag{8.16}$$

The values of the resonant frequency and damping factor are:

$$\omega_0 = \sqrt{K_1}\omega_n \tag{8.17}$$

$$\zeta = \frac{K_0}{\sqrt{K_1}} \tag{8.18}$$

where $\omega_n = \lambda_r / \sqrt{LJ}$ is the motor's natural resonant frequency. With the default values of K_0 and K_1 of 1 and 0.25, the system will be over-damped which is acceptable.

Feedback Analysis for Speed Control

Figure 8.11 presents block diagram of the system under speed control derived from Figure 5.2 in Chapter 5 which can be used for mid-speed feedback analysis.

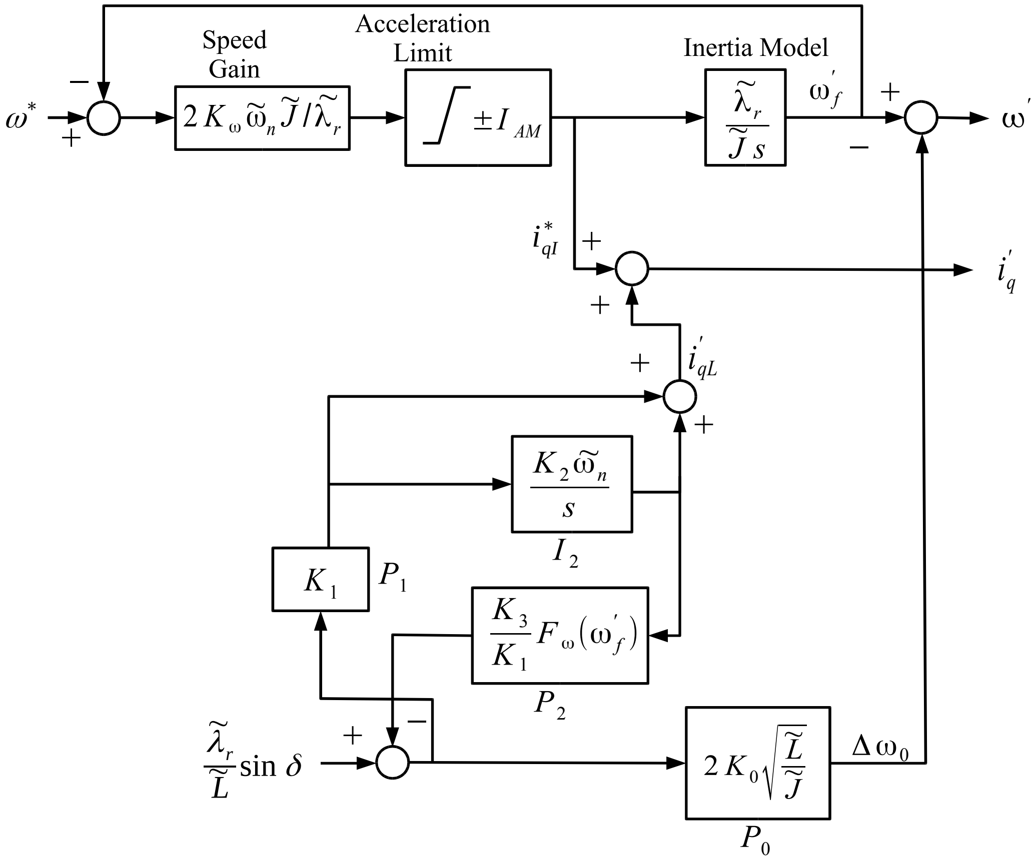

Figure 8.11. Feedback control block diagram under speed control.

In the feedback system of Figure 8.11, the P_2 block can be ignored as the function F_ω is zero in the mid-speed region. Also, the I_2 block can be ignored since K_2 is set so that this second order integral gain block has only a minor effect on stability. With these blocks ignored, sin δ approximated to δ in radians, and assuming the Feed Forward Converter keeps i_q equal to i'_q the feedback equation in Laplace notation can be derived as follows:

Using torque $T=\lambda_r i_q$, and assuming parameter estimates match actual parameters, the controller forces the torque to be:

$$T=\frac{K_1\lambda^2}{L}\delta \tag{8.19}$$

where δ is the measured flux angle error assumed here to be equal to the actual motor flux angle error.

Actual motor flux angle error is given by:

$$\delta=\frac{\omega'-\omega}{s} \tag{8.20}$$

Combine this with equation 8.19 to get:

$$\omega' - \omega = T \frac{L}{K_1 \lambda^2} s \tag{8.21}$$

This can be represented in a mechanical equivalent circuit as shown in Figure 8.12.

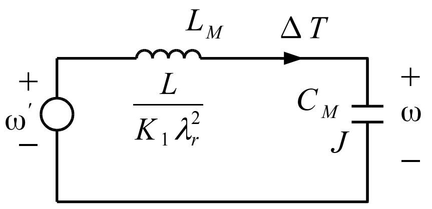

Figure 8.12. Mid-speed mechanical equivalent circuit showing effect of P_1.

The equation for the added damping block P0 is as follows:

$$\begin{aligned} \omega' &= \omega_f' - 2K_0 \sqrt{\frac{L}{J}} \cdot \frac{\lambda_r}{L} \delta \\ &= \omega_f' - \frac{2K_0}{K_1 \lambda_r} \sqrt{\frac{L}{J}} \; T \end{aligned} \tag{8.22}$$

This can be represented as a resistance in the mechanical equivalent circuit as shown in Figure 8.13.

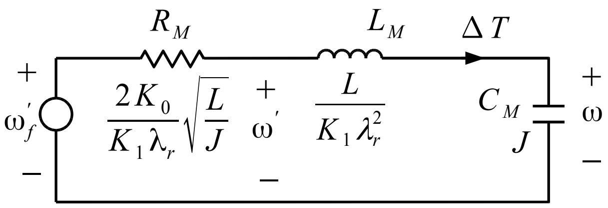

Figure 8.13. Mid-speed mechanical equivalent circuit including effect of P_0.

As for under torque control, the values of the resonant frequency and damping factor are:

$$\omega_0 = \sqrt{K_1} \omega_n \tag{8.23}$$

$$\zeta = \frac{K_0}{\sqrt{K_1}} \tag{8.24}$$

where $\omega_n = \lambda_r / \sqrt{LJ}$ is the motor's natural resonant frequency. With the default values of K_0 and K_1 of 1 and 0.25, the system will be over-damped which is acceptable.

The difference in the case of speed control is that motor, represented by CM in Figure 8.13, is now part of the feedback loop, making its operation dependent on the motor and load characteristics. An advantage of this is that any resonant in the motor coupling to the load, such as shown in Figure 8.7, will be at least partially damped by the feedback.

Low Speed Region

For sensorless control of a brushless DC motor, the most problematic area of operation is the low speed region. The standard method of operating in this region is to inject a current into the stator to lock up the rotor then to rotate the current around the windings to turn the rotor. The problem with this method is that there is no damping of the rotor allowing it to easily oscillate. These oscillations can build up to the point where rotor synchronisation is lost. The standard solution to this problem is to simply speed up the rotor rapidly to the point where normal operation with back emf detection can be commenced before oscillations have time to build up.

For example, Figure 8.14 shows the starting position of a brushless DC motor operated with a 6-step controller. A holding voltage is usually applied between two phases to generate a holding current I_h causing the rotor to lock in alignment with these phases. This voltage is then stepped around the phases causing the rotor to rotate.

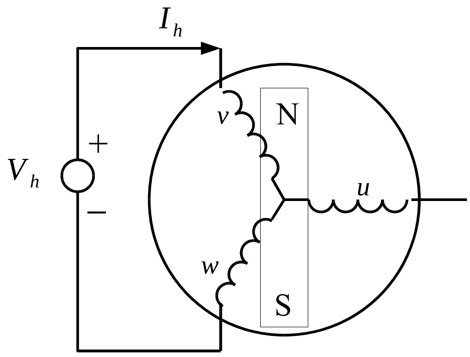

Figure 8.14. Start-up of a BLDC motor with 6-step controller.

Because the third phase is left unconnected, there is minimal damping of the rotor, allowing it to easily jump out of synchronisation as the rotor is rotated.

For a brushless DC motor operated with the more advanced field oriented control method, a similar starting method is used, with the difference being the holding current is injected using a current source created by current feedback around the inverter rather than using a fixed voltage to generate the current. The third phase stays connected to the inverter but is also connected as a current source using feedback with the injected current set to zero. As a result, the rotor is still undamped and the same start-up problems occur.

For a stepper motor, 6-step operation is used usually with the variation that instead of using a fixed voltage, a fixed current is injected into two phases using current feedback but with the third phase unconnected. One difference is that natural damping from friction and electromagnetic losses provide enough damping to provide stability. Also, the motor stays in this mode of operation over its full speed range with the injected current rotated to rotate the rotor at the desired speed. A large but accepted disadvantage of this method is that motor losses are high due to the usually large injected current required and that the motor can easily lose synchronism at higher speeds due to resonances strong enough to overcome the limited damping.

For the new controller, *d* axis current injection is also used to lock the rotor in position but with the crucial difference that stable operation is achieved by the simple procedure of controlling the impedance of the inverter output connected to the third phase. Adding the correct driving impedance to the third phase provides effective electrical damping to the rotor allowing mostly unrestricted low speed operation.

Feedback Analysis

Consider a two-phase, two-pole PMSM with no external load except the load inertia and with a quasi-stationary rotor with the controller *d* and *q* axes aligned with the two windings. Assume a fixed current I_d in the winding aligned with the *d* axis locking the rotor to this axis. Now consider the equivalent electrical *LC* circuit for the stationary *q* axis for an undamped oscillating rotor with the q axis winding open circuit. The equivalent *LC* tuned circuit is shown in Figure 8.4. The equivalent parallel inductance is L_P and the equivalent capacitance is C_P.

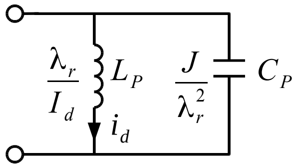

Figure 8.15. Quasi-stationary open circuit q axis equivalent circuit with fixed d axis current

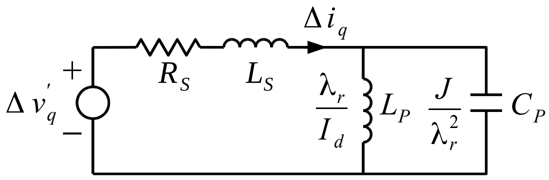

Figure 8.16. Complete q axis quasi-steady state equivalent circuit of the motor at zero speed.

To derive the values of L_P and C_P consider when the rotor is off the *d* axis current alignment by a small angle d during unstable oscillation. The resulting restoring torque is $I_d \lambda_r \sin\delta$ or $I_d \lambda_r \delta$ to a first approximation. The equation of motion of the rotor is:

$$J \frac{d^2\delta}{dt^2} = -I_d \lambda_r \delta \tag{8.25}$$

For electrical circuit equivalence, this must match the differential equation for the equivalent tuned circuit with circulating current i_P which is:

$$C_P \frac{d^2 i_P}{dt^2} = -\frac{1}{L_P} i_P \tag{8.26}$$

The tuned circuit component values can be found by matching terminal voltage, energy and resonant frequency. The q axis terminal voltage is $\lambda_r d\,\delta/dt$ which in the equivalent circuit is the voltage on the capacitor giving the energy stored in the capacitor as $0.5\,C_P(\lambda_r\; d\,\delta/dt)^2$. This is the kinetic energy $0.5\,J\,(d\,\delta/dt)^2$ giving:

$$C_P = \frac{J}{\lambda_r^2} \tag{8.27}$$

Matching the inverse square of the resonant frequencies gives $L_P C_P = J/(I_d \lambda_r)$ giving a value for the inductor of:

$$L_P = \frac{\lambda_r}{I_d} \tag{8.28}$$

Finally, equating the capacitor voltage $\lambda_r d\,\delta/dt$ to the inductor voltage $L_p di_p/dt$, the rotor deviation angle is found to be:

$$\delta = \frac{i_p}{I_d} \tag{8.29}$$

To complete the q axis equivalent circuit for quasi-stationary rotor oscillations, the motor stator winding inductance L_S and resistance R_S can be added as shown in Figure 8.16. Also added to this circuit is the q axis excitation voltage deviation $\Delta v'_q$ from this controller. The effective structure of the controller at zero speed, simplified from the actual controller structure showed in Figure 1.1 of Chapter 1 is shown in Figure 8.17.

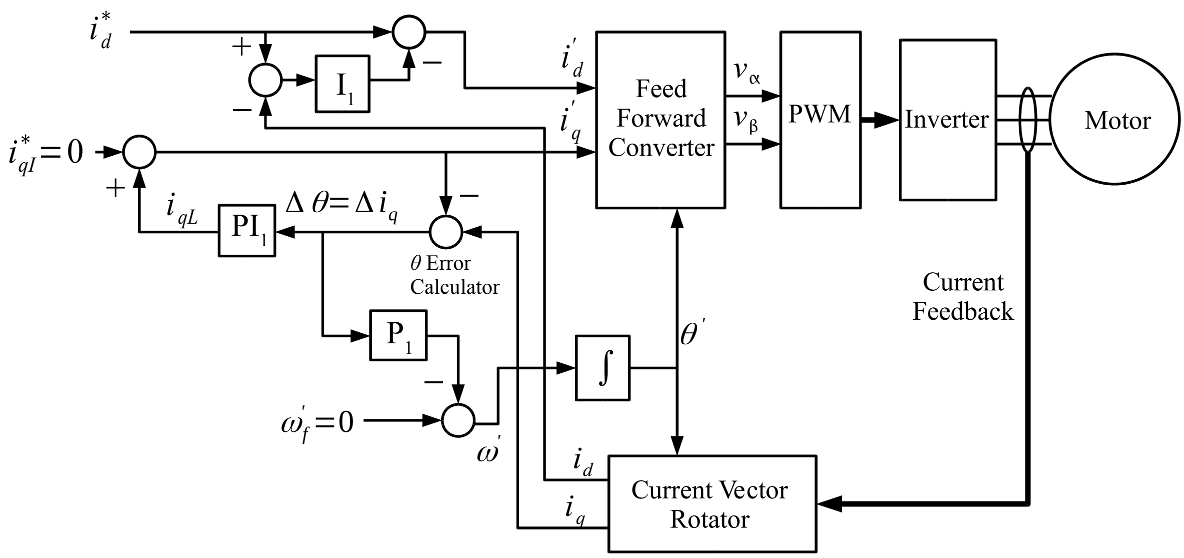

Figure 8.17. Effective controller structure at zero speed.

Note that the controller here presents a voltage source to the q axis for instability oscillations. At frequencies higher than the instability oscillations, the q axis impedance is modified by the presence of the PI_1 proportional integral feedback term in Figure 8.17. Other controllers present either an open circuit (six-step control) or a current source (field oriented control). This circuit model was derived for a stationary motor with no load but it also holds for low speed operation. $\Delta v'_q$ is controlled by the Feed Forward Converter block which connects it to changes in the applied speed according to the equation:

$$\Delta v'_q = \Delta \omega' \tilde{\lambda}_r \tag{8.30}$$

The equivalent circuit as shown in Figure 8.16 has a parallel tuned circuit $L_P C_P$ and a series tuned circuit $L_S C_P$ which set the minimum and maximum values of motor resistance R_S for good over-all damping.

For a parallel tuned circuit LC with a parallel damping resistance R, the damping factor ζ is given by:

$$\zeta = 0.5 \frac{\sqrt{L/C}}{R} \tag{31}$$

and for a series tuned circuit with a series damping resistor:

$$\zeta = 0.5 \frac{R}{\sqrt{L/C}} \tag{32}$$

To keep the damping factor for both tuned circuits greater than, say, 0.5 – a reasonable value for slight under-damping, resistance R must be kept between $\sqrt{L_S/C_P}$ and $\sqrt{L_P/C_P}$. If the motor resistance is not enough to provide the required damping, virtual resistance can be added to the inverter output impedance via feedback. A negative virtual resistance can also be added to the inverter if the motor resistance is too large but this is not recommended as it can easy lead to controller instability.

As well as controlling the q axis inverter resistance, there is one other effect that the controller incorporates to improve damping at zero and low speed. This is the movement of the current vector angle in response to changes in the q axis current. As can be seen in Figure 8.17. changes in the q axis current causes modulation of the applied speed ω' and, through integration, the applied rotor angle θ' via the proportional constant P_1. The Feed Forward Converter block, described in detail in Chapter 3, rotates the angle of the current vector applied to the motor instantaneously in response to a change in θ'. The effect of this on the equivalent circuit representation of the controller and motor shown in Figure 8.16 is to change the restoring torque to neutral position applied to the rotor and thus its potential energy as represented in the energy stored in the capacitor C_P. The equivalent circuit of Figure 8.16 modified to incorporate this effect is shown in Figure 8.18. In this circuit the secondary effect of the current feedback term i_{qL} is ignored resulting in $\Delta i_q = i_q$. The constant K_0 is the damping factor as described in Chapter 5 and is usually set to 1. Also shown in the equivalent circuit is the effect of a torque disturbance ΔT_L injecting a current $I_L = \Delta T_L / \lambda_r$.

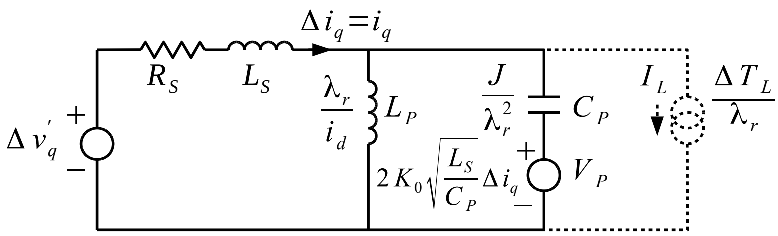

Figure 8.18. Zero speed equivalent circuit incorporating controller stability correction.